COT 6410: Pipeline Scheduling

Michael Gabilondo

Overview

Real-World Problem

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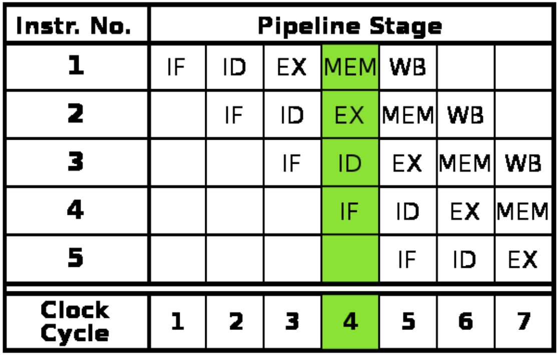
- I chose a real-world problem: **Instruction Scheduling on a Pipeline with precedence constraints between pipeline stages of jobs**. or, PIPELINE-SCHEDULING for short.
- It's this problem, from your Computer Architecture course:



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3			IF	ID	EX	MEM	WB
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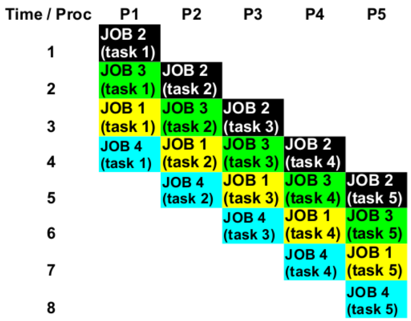
- The IBM System/360, which implemented Tomasulo's algorithm, is a notable example of a dynamic scheduling processor [Hennessy, "Computer Architecture"].

A different view of the pipeline chart



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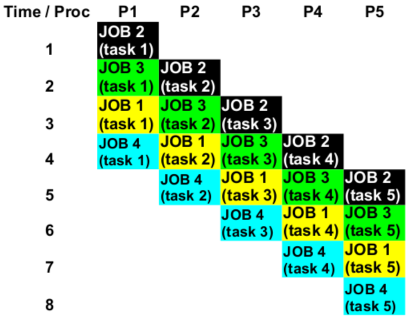

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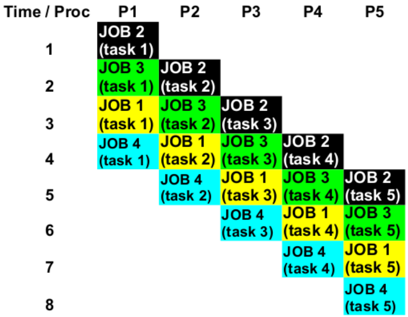

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- Each Instruction/Job has m tasks, the number of pipeline stages. Here, m = 5.

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- This appears to be a **shop-scheduling problem**.

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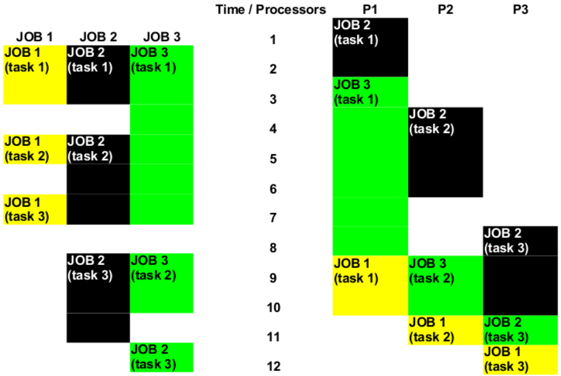
- Tasks have a length, $l(t)$; if a task t on processor i is scheduled on $\sigma_i(t)$, then nothing else can be scheduled on that processor until $\sigma_i(t) + l(t)$

Flow-Shop Scheduling

- QUESTION: Can the tasks be scheduled on the processors such that the above constraints are obeyed and the finishing time of the last task is less than D?



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- A partial order $\prec$ is defined over the tasks. $t \prec t'$ means that task t' cannot begin until task t finishes because, for example, t produces data that is required by t'.

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  - An integer D, the deadline for the last task.
- QUESTION:
  - Is there a schedule with the finishing time of the last task less than D, and the schedule is a **no-wait flow-shop schedule** and if $t_{i'}(j') \prec t_i(j)$, then $\sigma_{i'}(t_{i'}(j')) < \sigma_i(t_i(j))$?

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- I found another problem which represents the pipeline problem more closely.

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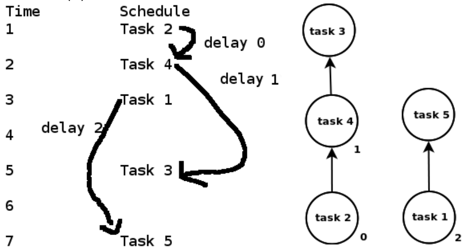
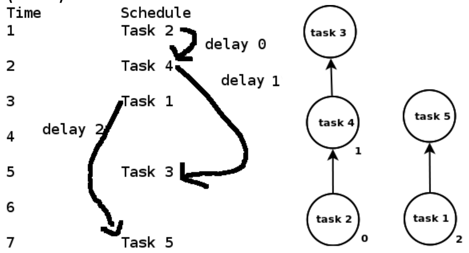


MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS

- **QUESTION**: Is there a one-processor schedule S for T that obeys the precedence constraints and the delays, and the maximum $S(t) \leq D$? The schedule S is an injective function $S : T \to Z^+$ such that, for each edge $\langle t_i, t_j \rangle \in E, S(t_j) - S(t_i) > d(t_i) \Leftrightarrow S(t_j) \geq S(t_i) + d(t_i) + 1$



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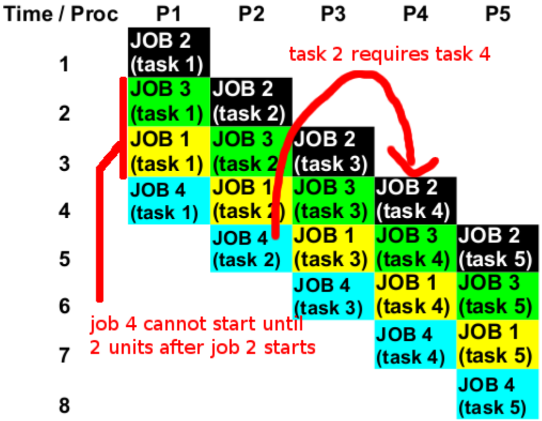
- My transformation created sets of elements with D or $d(t)$ elements, so it is not a polynomial transformation
- I modified pipeline scheduling so that sets of tasks were not part of the problem, and modified the precedence constraints; I will show how this new formulation is very similar to the old problem

Redefining Pipeline Scheduling

- What is the effect of the precedence constraints between tasks on the starting times of the first task of each job?



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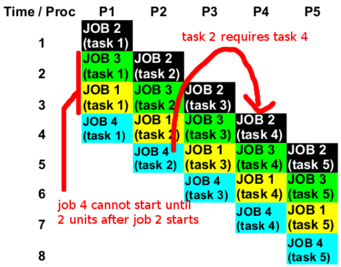

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job 4 cannot start until 2 units after job 2 starts

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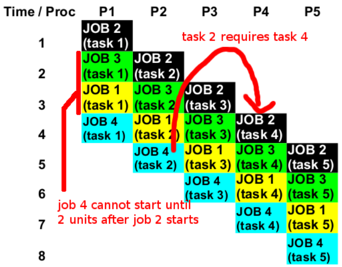

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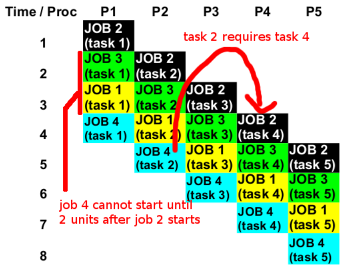

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- If $t_k(j') < t_i(j)$ and $k < i$, then the first task task of job j must scheduled at least one time unit after job $t_1(j')$ is scheduled.
- Also, the maximum "delay" a precedence constraint can have is $k - i + 1 \leq m$.

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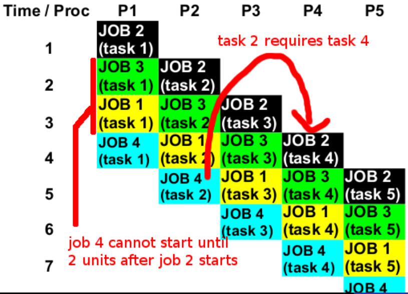
- $\langle t_i, t_j \rangle \in E \Leftrightarrow S(t_j) \geq S(t_i) + W(\langle t_i, t_j \rangle) + 1$
  In this case, we put an edge between Job 4 and Job 2 with weight 2.



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- For the maximum $\sigma(I)$, does $\sigma(I) + (m-1) \leq D$?

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- Accept an instance of MINIMUM PRECEDENCE CONSTRAINED SEQUENCING WITH DELAYS.
- INSTANCE: Set T of tasks, a directed acyclic graph $G = (T, E)$ defining precedence constraints for the tasks, a positive integer D, and for each task an integer delay $0 \leq d(t) \leq D$.
- QUESTION: Is there a one-processor schedule S for T that obeys the precedence constraints and the delays, and the maximum $S(t) \leq D$? The schedule S is an injective function $S : T \rightarrow Z^+$ such that, for each edge $\left\langle t_i, t_j \right\rangle \in E, S(t_j) - S(t_i) > d(t_i) \Leftrightarrow S(t_j) \geq S(t_i) + d(t_i) + 1$

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- Solve PIPELINE SEQUENCING and return the answer
- The time complexity is $O(T+E)$; this is a polynomial transformation.

Proof that Pipeline Sequencing is NP-Complete

- Notice that if $\left\langle t_i, t_j \right\rangle \in E$, then $S(t_j) \geq S(t_i) + d(t_i) + 1$. The equivalent condition can hold in PIPELINE SEQUENCING by adding that edge to E' and and setting the weight on that edge to $d(t_i)$.

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- The respective instances and constraints of the two problems have been constructed to be equivalent:
- If the SEQUENCING WITH DELAYS instance is true and the last job is scheduled at time t, then the last job of PIPELINE SEQUENCING will also be scheduled at time t. The PIPELINE SEQUENCING instance adds $m-1$ time units to the finishing time t, so we set D' to $D+(m-1)$.

Proof that Pipeline Sequencing is NP-Complete

- If the PIPELINE SEQUENCING instance is true, then it has a schedule that meets its constraints; if the last job of the PIPELINE SEQUENCING instance is scheduled at time t, then so will the corresponding job in SEQUENCING WITH DELAYS also be scheduled at time t. In PIPELINE SEQUENCING, the problem ends $m+1$ time units after the last job; this is why $m+1$ time units were added to the deadline D when constructing the PIPELINE SEQUENCING instance.

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- Thus, the constructed instance is true if and only if the base instance is true.
- Now, once I show Pipeline Sequencing is in NP, then that proves it is NP-Complete.

Pipeline Sequencing is in NP

- Is PIPELINE SEQUENCING in NP? An oracle can provide the starting time for each of the instructions, and an algorithm simply needs to check the DAG to see if the schedule is valid. For each edge in the DAG, the algorithm simply needs to check if the starting time of the dependant task is large enough and comes after the the first task; it also needs to check the deadline is met by simply checking the starting time of the last instruction. So the problem is NP.

Conclusion

- This has show that the real-life problem of **Instruction Scheduling on a Pipeline with precedence constraints between pipeline stages of jobs** is NP-Hard for an **unbounded** number of processors. In practice, this result does not apply because most machines have a small number of pipeline stages.
- We were unable to show that the **pipeline scheduling problem with sets of m tasks** was NP-Complete because our **polynomial transormation** was creating instances where m was exponential. We have not shown it is not NP-complete, however; we don't know.

Conclusion

- But, I do not think it is NP-Complete, because the paper by Hennesy showed that the SEQUENCING WITH DELAYS problem was NP-Complete for **unbounded** m. They had to introduce extra constraints into the problem to get a bound on m.

References

- "A compendium of NP optimization problems", Pierluigi Crescenzi and Viggo Kann. 1998.
- "Computers and Intractability: A Guide to the Theory of NP-Completeness", Garey and Johnson. 1979.
- "Postpass Code Optimization of Pipeline Constraints", Hennessy, John L. and Gross, Thomas. 1983.
- "The Three-Machine No-Wait Flow Shop is NP-Complete", Hans Röck. 1984.