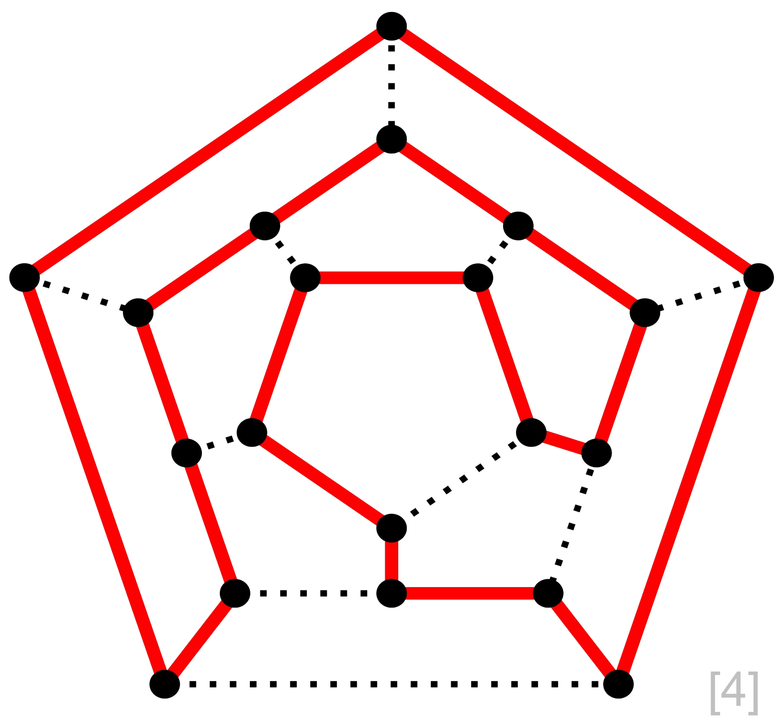
Hamiltonian Paths/Cycles
Eulerian Paths/Cycles

Slides prepared for UCF's COP 3503 recitation by Michael Gabilondo on October 23, 2013

Hamiltonian path/cycle

- **<u>Hamiltonian path</u>** – "a **path** in an undirected or directed graph that visits each **vertex** exactly once" [1]
- **<u>Hamiltonian cycle</u>** – "a Hamiltonian path that is a cycle." [1] *(The first and last vertices are the same)*



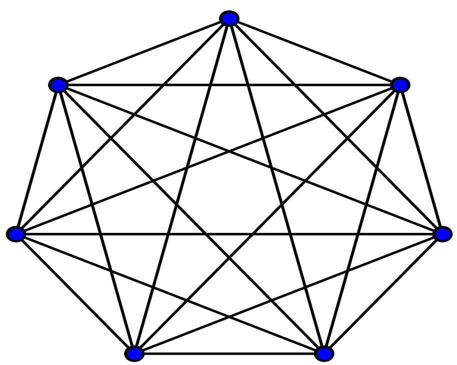
[4]

"A Hamiltonian cycle in a dodecahedron. Like all platonic solids [considered as graphs], the dodecahedron is Hamiltonian." [1]

(There are four other platonic solids: Tetrahedron, Cube or Hexahedron, Octahedron, Icosahedron [7])

Examples of Graphs with Hamiltonian cycles

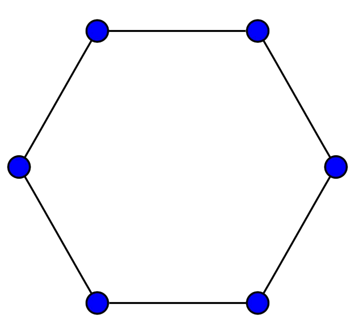
- "A **complete graph** with more than two vertices is Hamiltonian" [1]



The complete graph, K7.

It contains n = 7 vertices and n(n – 1)/2 edges.

- "Every **cycle graph** is Hamiltonian" [1]



The cycle graph, C6.

It contains n = 6 vertices and n = 6 edges.

Hamiltonian Path/Cycle problems

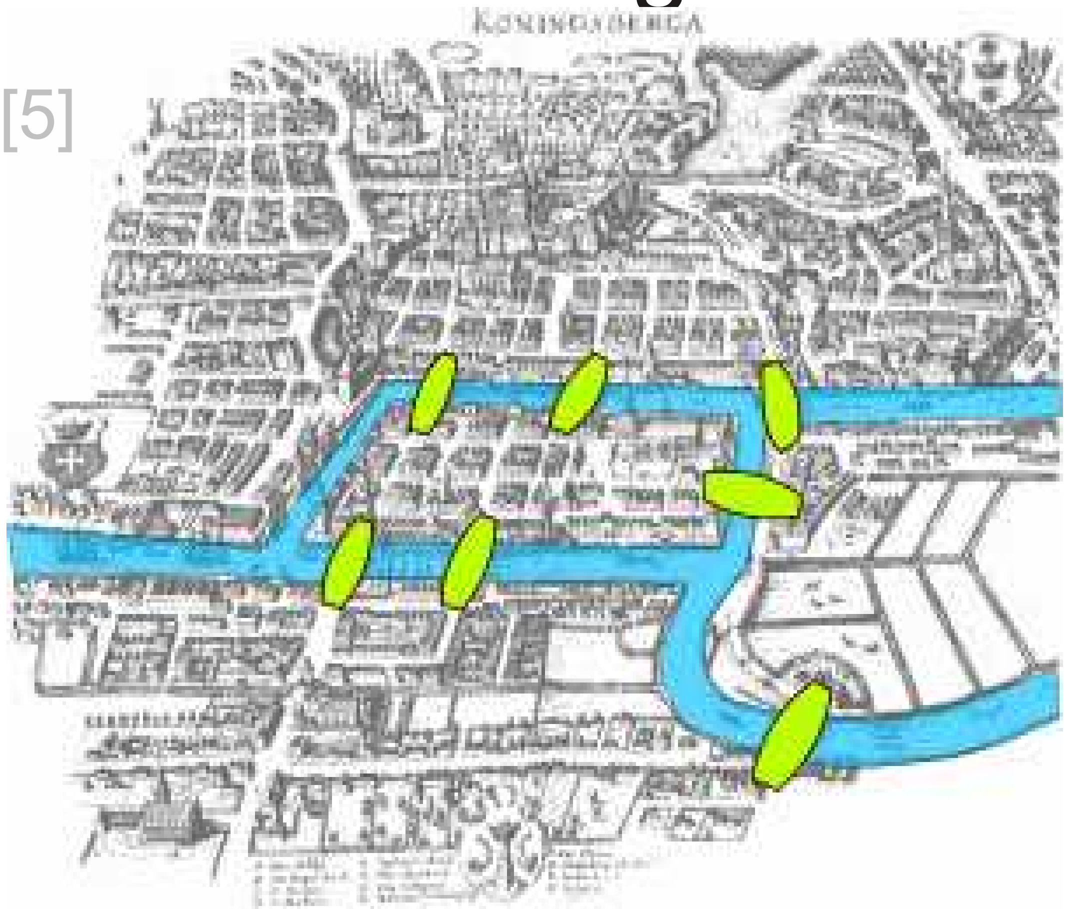
- **Hamiltonian path problem** – Does the graph have a Hamiltonian path?
- **Hamiltonian cycle problem** – Does the graph have a Hamiltonian cycle?
- These problems are NP-Complete [1]
  - A class of decision problems. Any instance of any problem can be transformed into an instance of any other problem by a polynomial-time algorithm. Therefore, if one problem has a polynomial-time solution, then they all do. Just transform instances of the problem into the new problem and run the polynomial-time solution for that problem. No such polynomial-time solution is known.
- In a graph of n vertices, there are n! paths that *could* be Hamiltonian paths/cycles [8]
    - Start at a vertex 1,2,...,n. For each of those, there are n-1 remaining ways to extend the path. For each of those ways, there are n-2 ways left...
- Since these problems are NP-complete, we know there is no known polynomial-time algorithm to solve the problem for arbitrary graphs
  - However, this does not exclude the possibility that there may exist faster approximation methods (which may return a wrong answer with a small probability), and faster exact methods for restricted classes of graphs

The Traveling Salesman Problem

- **The Traveling Salesman problem -** Given a graph with weighted edges, find a cycle that visits all vertices exactly once whose sum of edge weights is minimal among all cycles that visit all vertices.
- The above is a function problem, not a decision problem
- The decision problem version just asks whether such a cycle exists (whose sum of edge weights is less than k, or is equal to k, etc.)
- The decision versions of Hamiltonian Cycle and Traveling Salesman are NP-Complete. (The function versions are NP-hard.)
- The Hamiltonian cycle problem is a special case of The Traveling Salesman problem. That means we can solve Hamiltonian Cycle by making an instance of Traveling Salesman
  - Input to transformation algorithm: an instance of Hamiltonian Cycle.
  - Make a new complete weighted graph. Set the edge weight to 1 if the vertices are adjacent in the original graph. Set the edge weight to 2, otherwise.
  - Run the algorithm for Traveling Salesman: If there exists a cycle (of n vertices) whose sum of weights is n, then there exists a Hamiltonian Cycle in the original graph. [8]

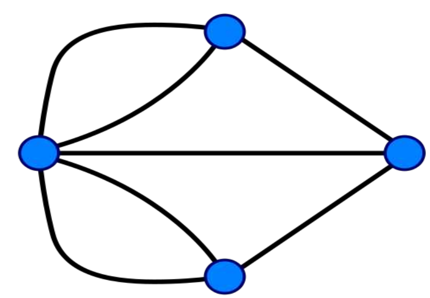
Introduction to Eulerian Paths/Cycles – Seven Bridges of Königsberg

[5]



Problem:

"Find a walk* through the city that would cross each bridge once and only once" [4]

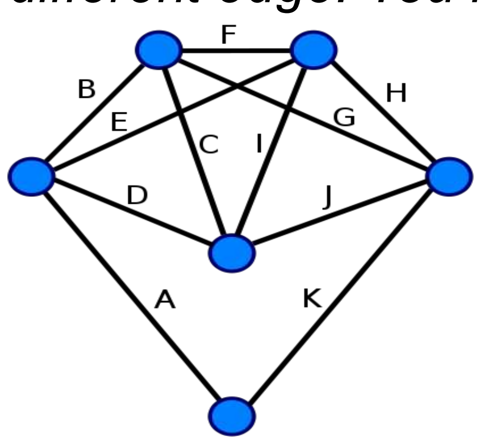


[6]

* A **walk** refers to a general kind of **path** whose vertices can be repeated, and whose first and last vertices can be the same or not. (A path can be used to mean the same thing as a walk.)
Since the edges cannot be repeated, this particular kind of walk is also a **trail**.

Eulerian Paths/Cycles

- **<u>Eulerian path/trail</u>** – “a trail in a graph which visits every edge exactly once” [2] *(trail = walk/path with no repeated edges)*
- **<u>Eulerian cycle/circuit</u>** – “an Eulerian trail which starts and ends on the same vertex” [2] *(circuit = trail with same start/end vertices)*
- “For the existence of Eulerian trails it is necessary that no more than two vertices have an odd degree” [2] (0, 1, or 2 vertices can have odd degree)
- If there are no vertices of odd degree, all Eulerian trails are circuits. [2]
  - *Every time you visit a vertex by one edge, you can leave it by a different edge. You have to start and end at the odd-degree vertices.*



“Every vertex of this graph has an even degree, therefore this is an Eulerian graph. Following the edges in alphabetical order gives an Eulerian circuit/cycle.” [2]

Fleury's algorithm for finding Euler paths/cycles

Quoted verbatim from [2]

- Consider a graph known to have all edges in the same component and at most two vertices of odd degree
- The algorithm starts at a vertex of odd degree, or, if the graph has none, it starts with an arbitrarily chosen vertex.
- At each step it chooses the next edge in the path to be one whose deletion would not disconnect the graph, unless there is no such edge, in which case it picks the remaining edge left at the current vertex.
- It then moves to the other endpoint of that vertex and deletes the chosen edge.
- At the end of the algorithm there are no edges left, and the sequence from which the edges were chosen forms an Eulerian cycle if the graph has no vertices of odd degree, or an Eulerian trail if there are exactly two vertices of odd degree.

- The graph traversal considers O(|E|) edges. Detecting bridges (edges which would disconnect the graph) when choosing the next edge using Tarjan's linear bridge-finding algorithm makes the Fleury's algorithm O(|E|^2). There are faster algorithms. [2]
- Notice that a polynomial-time algorithm exists for finding Euler paths/cycles, but one does not exist for finding Hamiltonian paths/cycles.

Properties of Eulerian Paths/Cycles

- "An undirected graph has an Eulerian cycle if and only if every vertex has even degree, and all of its vertices with nonzero degree belong to a single connected component." [2]
- "An undirected graph has an Eulerian trail if and only if at most two vertices have odd degree, and if all of its vertices with nonzero degree belong to a single connected component." [2]
  - If there are exactly two vertices of odd degree, all Eulerian trails start at one of them and end at the other. " [2]
- "A directed graph has an Eulerian cycle if and only if every vertex has equal in degree and out degree, and all of its vertices with nonzero degree belong to a single strongly connected component." [2]

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